A Multi-Line TRL Calibration

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A structure with c and π modes of propagation and corresponding characteristic impedances has a transmission matrix defined as

$$\begin{bmatrix} b_c^1 \\ b_\pi^1 \\ a_c^1 \\ a_\pi^1 \end{bmatrix} = [T] \cdot \begin{bmatrix} a_c^2 \\ a_\pi^2 \\ b_c^2 \\ b_\pi^2 \end{bmatrix} \tag{1}$$

with the T-parameters L of a line of length l given by

$$[L] = \begin{bmatrix} e^{-\gamma_c l} & 0 & 0 & 0 \\ 0 & e^{-\gamma_\pi l} & 0 & 0 \\ 0 & 0 & e^{+\gamma_c l} & 0 \\ 0 & 0 & 0 & e^{+\gamma_\pi l} \end{bmatrix}. \tag{2}$$

If a measurement system is created with independent 4-port error boxes with transmission parameters A and B, then the measurement of the line is given by the cascade

$$M_L = A \cdot L \cdot B \tag{3}$$

and the measurement for a thru connection given by

$$M_T = A \cdot B. \tag{4}$$

Following the derivation for the single line TRL calibration, the measurement quantity

$$Q \equiv M_L \cdot M_T^{-1} = (A \cdot L \cdot B)(B^{-1} \cdot A^{-1}) = A \cdot L \cdot A^{-1} \tag{5}$$

is formed. This form defines Q as a similarity transform of L. Similar matrices have equal eigenvalues so the eigenvalues of Q provide the modal propagation terms.

$$\begin{gathered} L_{ii} = eigenvalues(Q) \\ \det\left[Q - L_{ii} \cdot I\right] = 0 \end{gathered} \tag{6}$$

and may be determined using many standard algorithms.

The matrix A consists of the eigenvectors associated with the similarity transform for each eigenvalue which are given by the indeterminate equation

$$\left[Q - L_{ii} \cdot I\right] \cdot \begin{bmatrix} A_{1i} \\ A_{2i} \\ A_{3i} \\ A_{4i} \end{bmatrix} = 0 \tag{7}$$

which can be solved for the normalized eigenvectors

$$\begin{bmatrix} \dfrac{A_{1i}}{A_{4i}} \\ \dfrac{A_{2i}}{A_{4i}} \\ \dfrac{A_{3i}}{A_{4i}} \end{bmatrix} = \begin{bmatrix} Q_{11} - L_{ii} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} - L_{ii} & Q_{23} \\ Q_{311} & Q_{32} & Q_{33} - L_{ii} \end{bmatrix} \cdot \begin{bmatrix} -Q_{14} \\ -Q_{24} \\ -Q_{34} \end{bmatrix} \tag{8}$$

leaving only the A_{4i} unknown in A. A is then separated into known, $\underline{A}$, and unknown $\underline{\underline{A}}$ parts,

$$A = \underline{A}\cdot\underline{\underline{A}} = \begin{bmatrix} \frac{A_{11}}{A_{41}} & \frac{A_{12}}{A_{42}} & \frac{A_{13}}{A_{43}} & \frac{A_{14}}{A_{44}} \\ \frac{A_{21}}{A_{41}} & \frac{A_{22}}{A_{42}} & \frac{A_{23}}{A_{43}} & \frac{A_{24}}{A_{44}} \\ \frac{A_{31}}{A_{41}} & \frac{A_{32}}{A_{42}} & \frac{A_{33}}{A_{43}} & \frac{A_{34}}{A_{44}} \\ 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} A_{41} & 0 & 0 & 0 \\ 0 & A_{42} & 0 & 0 \\ 0 & 0 & A_{43} & 0 \\ 0 & 0 & 0 & A_{44} \end{bmatrix}. \tag{9}$$

Similarly, the measurement quantity R may be defined

$$R \equiv M_T^{-1} M_L = (B^{-1}A^{-1})(ALB) = B^{-1}LB = CLC^{-1}, \tag{10}$$

where C is the inverse of B. The eigenvectors $\underline{C}$ and unknown C_{4i} terms contained in $\underline{\underline{C}}$,

$$C \equiv B^{-1} = \underline{C}\cdot\underline{\underline{C}} \tag{11}$$

are determined in the same way as the elements of A using (8) with Q replaced by R.

The thru measurement relates the unknown terms of A and C

$$A = M_T \cdot B^{-1} = M_T \cdot C = M_T \cdot \underline{C}\cdot\underline{\underline{C}} = \underline{A}\cdot\underline{\underline{A}}, \tag{12}$$

so

$$\underline{\underline{A}}\cdot\underline{\underline{C}}^{-1} = \underline{A}^{-1}\cdot M_T \cdot \underline{C} \equiv Y \tag{13}$$

where Y is diagonal with known terms equal to the ratios of the unknown terms in A and C,

$$Y_{11} = \frac{A_{41}}{C_{41}} \quad Y_{22} = \frac{A_{42}}{C_{42}} \quad Y_{33} = \frac{A_{43}}{C_{43}} \quad Y_{44} = \frac{A_{44}}{C_{44}}. \tag{14}$$

The eigenvalues, the normalized eigenvectors, and the ratios between the magnitudes of the eigenvectors is all that can be determined from the thru and line standards. Again, following the single line TRL calibration, a reflect standard is required. This two-port standard need not be fully known; however, some information analagous to the sign choice in the single line case is required.

The one-port measurement of the reflect standard for the single line case becomes a two-port measurement (as opposed to four) containing only the terms of error box A. Using the definition of the transmission parameters the scattering parameters of this measurement, w_A, may be written in terms of the error box transmission parameters and the scattering parameters of the unknown reflect, R. In the equation

$$w_A = \left(A_1 \cdot R + A_2\right)\cdot\left(A_3 \cdot R + A_4\right)^{-1}, \tag{15}$$

the A_i are two-by-two submatrices of the A matrix defined by

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = \begin{bmatrix} \underline{A}_1 & \underline{A}_2 \\ \underline{A}_3 & \underline{A}_4 \end{bmatrix} \cdot \begin{bmatrix} \underline{\underline{A}}_1 & 0 \\ 0 & \underline{\underline{A}}_2 \end{bmatrix} \tag{16}$$

with similar definitions for the $\underline{A}_i$ and diagonal $\underline{\underline{A}}_i$ terms.

Expressing and reorganizing (15) in terms of the $\underline{A}_i$ and $\underline{\underline{A}}_i$ provides the relation

$$\underline{\underline{A}}_1 \cdot R = \left(w_A \cdot \underline{A}_3 - \underline{A}_1\right)^{-1} \cdot \left(\underline{A}_2 - w_A \cdot \underline{A}_4\right) \cdot \underline{\underline{A}}_2 = Z \cdot \underline{\underline{A}}_2 \tag{17}$$

where Z is defined by

$$Z \equiv \left(w_A \cdot \underline{A}_3 - \underline{A}_1\right)^{-1} \cdot \left(\underline{A}_2 - w_A \cdot \underline{A}_4\right) \tag{18}$$

and is fully known. Equation (17) provides an indeterminate set of equations for the elements of $\underline{\underline{A}}$, allowing the determination of the A_{4i}/A_{44} ratios in terms of the known Z and unknown R:

$$\frac{A_{41}}{A_{44}} = \frac{Z_{12}}{R_{12}} \quad \frac{A_{42}}{A_{44}} = \frac{Z_{22}}{R_{22}} \quad \frac{A_{43}}{A_{44}} = \frac{R_{21}}{R_{22}}\frac{Z_{22}}{Z_{21}} = \frac{R_{11}}{R_{12}}\frac{Z_{12}}{Z_{11}}. \tag{19}$$

A set of similar equations is developed for the reflect standard measured with error box B. The expression for the measured scattering parameters is different from (15) since the transmission matrix has a left-to-right orientation. The second reflect measurement is given by

$$W_B = \left(B_4 - R\cdot B_2\right)^{-1}\cdot\left(R\cdot B_1 - B_3\right) \tag{20}$$

where the B_i are submatrices as described for the previous case. The $\underline{\underline{B}}_i$ terms are then related by

$$\underline{\underline{B}}_2\cdot X = R\cdot \underline{\underline{B}}_1 \tag{21}$$

with X defined as

$$X = \left(\underline{B}_4\cdot W_B + \underline{B}_3\right)\cdot\left(\underline{B}_1 + \underline{B}_2\cdot W_B\right)^{-1} \tag{22}$$

which is fully known since B and C are inverse related:

$$B = C^{-1} = (\underline{C}\cdot\underline{\underline{C}})^{-1} = \underline{\underline{C}}^{-1}\cdot\underline{C}^{-1} \equiv \underline{\underline{B}}\cdot\underline{B}. \tag{23}$$

The indeterminate equations for the elements of the diagonal $\underline{\underline{B}}$ given by (21) determine the ratios:

$$\frac{C_{41}}{C_{44}} = \frac{R_{21}}{X_{21}} \quad \frac{C_{42}}{C_{44}} = \frac{R_{22}}{X_{22}} \quad \frac{C_{43}}{C_{44}} = \frac{R_{21}}{R_{11}}\frac{X_{11}}{X_{21}} = \frac{R_{22}}{R_{12}}\frac{X_{12}}{X_{22}}. \tag{24}$$

Combining (14), (19), and (24) by forming ratios of Y_{ii}/Y_{44} eliminates the unknowns yielding equations relating the terms of R to the elements of X, Y, and Z which are known. In particular,

$$R_{11}\cdot R_{11} = \frac{X_{11}\cdot Y_{33}\cdot Z_{11}}{Y_{11}}, \tag{25}$$

$$R_{22}\cdot R_{22} = \frac{X_{22}\cdot Y_{44}\cdot Z_{22}}{Y_{22}}, \tag{26}$$

and

$$R_{12}\cdot R_{21} = \frac{X_{21}\cdot Y_{44}\cdot Z_{12}}{Y_{11}}. \tag{27}$$

If a proper sign choice can be made in (25) and (26) and a known relationship exists between R_{12} and R_{21}, then the scattering parameters of the reflect standard have been fully determined. Using the terms of R in (14), (19), and (24) fully determines A and C (and therefore B) in terms of A_{44} which remains unknown. This is enough information for calibration. The measured transmission parameters for a device with unknown transmission parameters U is given by

$$M_U = A\cdot U\cdot B \tag{28}$$

and when combined with the thru measurement of (4) provides the result:

$$M_U\cdot M_T^{-1} = A\cdot U\cdot A^{-1} \tag{29}$$

The solution for U is then obtained from

$$U = A^{-1}\cdot M_U\cdot M_T^{-1}\cdot A = \frac{1}{A_{44}}\cdot\left(\frac{1}{A_{44}}\cdot A\right)^{-1}\cdot M_U\cdot M_T^{-1}\cdot A_{44}\cdot\left(\frac{1}{A_{44}}\cdot A\right) \tag{30}$$

which reduces (after eliminating the scalar terms) to a form containing only the known, normalized A matrix terms:

$$U = \left(\frac{1}{A_{44}}\cdot A\right)^{-1}\cdot M_U\cdot M_T^{-1}\cdot\left(\frac{1}{A_{44}}\cdot A\right). \tag{31}$$

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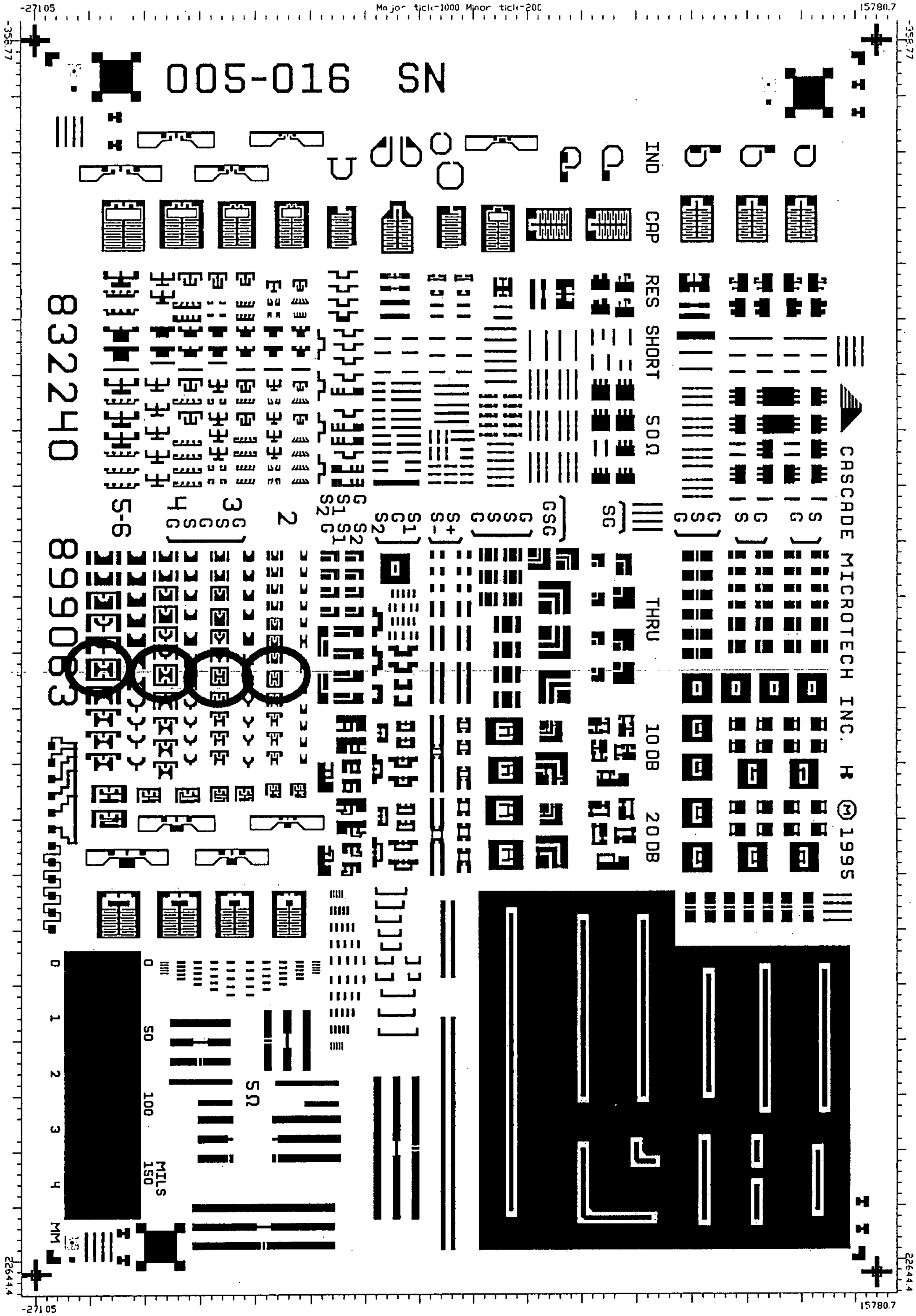

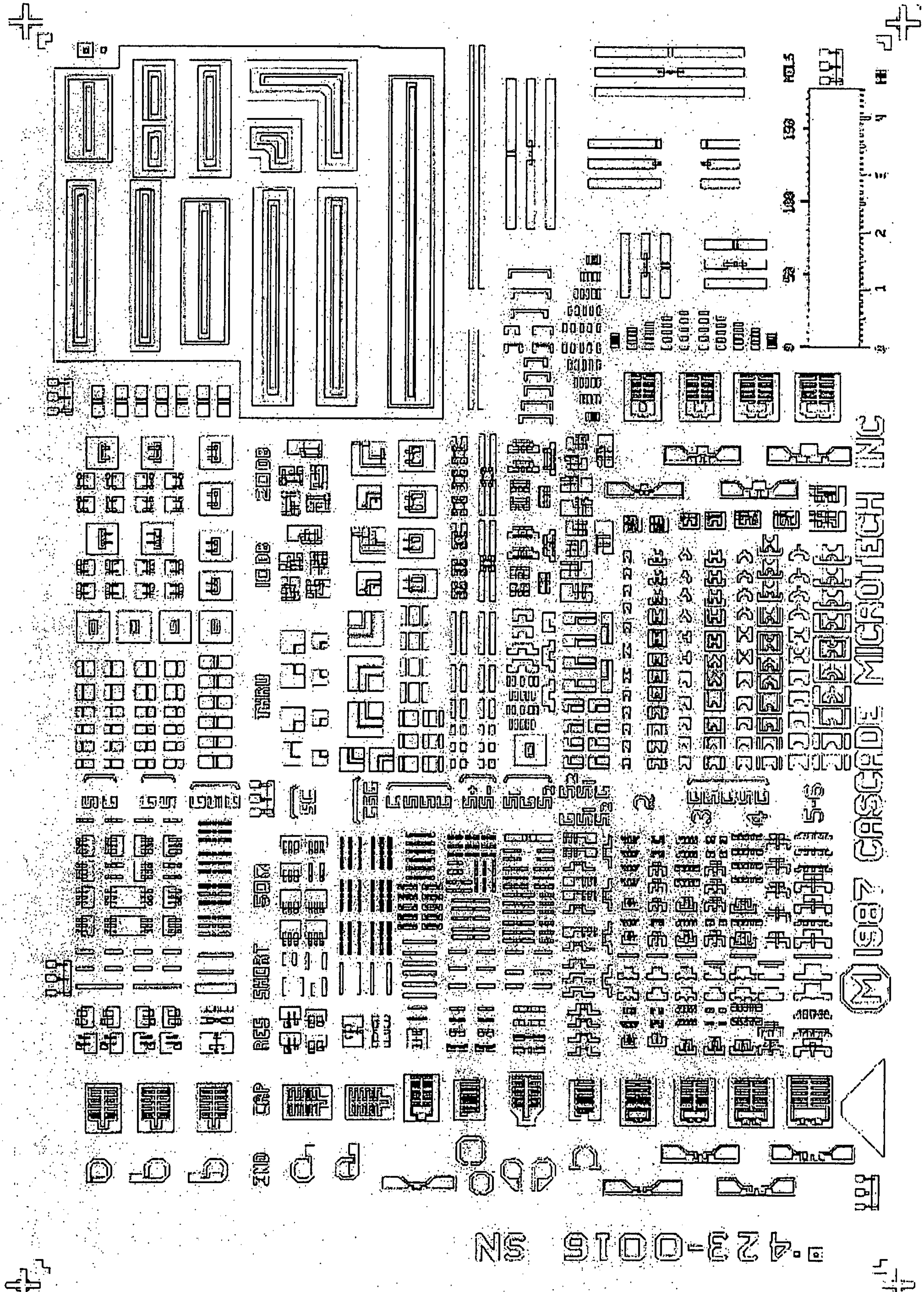

GENERAL PURPOSE